Probability and Random Processes (Part – I)

1. The variance of a random variable X is σ_x^2. Then the variance of –kX (where k is a positive constant) is
   (a) σ_x^2
   (b) $-k\sigma_x^2$
   (c) $k\sigma_x^2$
   (d) $k^2\sigma_x^2$

[GATE 1987: 2 Marks]

Soln. $\boldsymbol{Var(-kX) = E[(-kX)^2]}$

$\boldsymbol{\sigma^2 = E[k^2X^2]}$

$\boldsymbol{= k^2E[X^2]}$

$\boldsymbol{= k^2\sigma_x^2}$

Option (d)

2. White Gaussian noise is passed through a linear narrow band filter. The probability density function of the envelope of the noise at the filter output is
   (a) Uniform
   (b) Poisson
   (c) Gaussian
   (d) Rayleigh

[GATE 1987: 2 Marks]

Soln. The narrow band representation of noise is

$\boldsymbol{n(t) = n_c(t)\cos\omega_c t + n_s(t)\sin\omega_c t}$

Its envelope is $\boldsymbol{\sqrt{n_c^2(t) + n_s^2(t)}}$

$\boldsymbol{n_c(t)}$ ***and*** $\boldsymbol{n_s(t)}$ **are two independent zero mean Gaussian processes with same variance. The resulting envelope is Rayleigh variable**

Option (d)

3. Events A and B are mutually exclusive and have nonzero probability. Which of the following statement(s) are true?
   (a) $P(A \cup B) = P(A) + P(B)$
   (b) $P(B^C) > P(A)$
   (c) $P(A \cap B) = P(A)P(B)$
   (d) $P(B^C) < P(A)$

[GATE 1988: 2 Marks]

Soln. For mutually exclusive events A and B

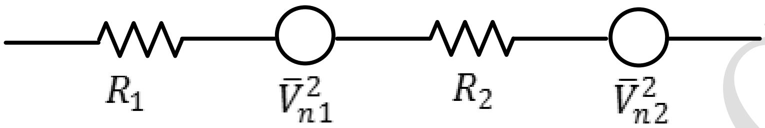
$$\boldsymbol{P(A \cup B) = P(A) + P(B)}$$

Option (a)

4. Two resistors R_1 and R_2 (in ohms) at temperatures $T_1^0\ K\ and\ T_2^0\ K$ respectively, are connected in series. Their equivalent noise temperature is ________ ^{0}K.

[GATE 1991: 2 Marks]

Soln.



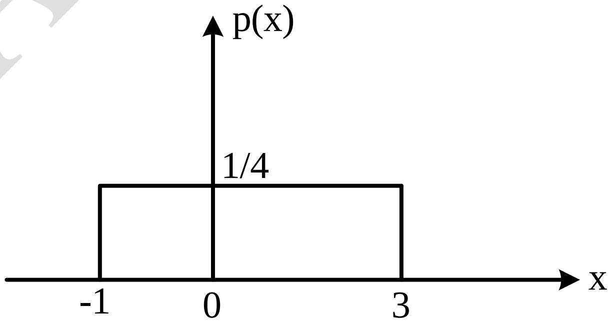
$$\boldsymbol{\overline{V}_n^2 = \overline{V}_{n_1}^2 + \overline{V}_{n_2}^2}$$

$$\boldsymbol{4KT_eB(R_1 + R_2) = 4KT_1BR_1 + 4KT_2BR_2}$$

$$\boldsymbol{T_e(R_1 + R_2) = R_1T_1 + R_2T_2}$$

$\boldsymbol{or\ \ T_e = (R_1T_1 + R_2T_2)/(R_1 + R_2)}$ **Equivalent noise temperature**

5. For a random variable 'X' following the probability density function, p(x), shown in figure, the mean and the variance are, respectively



(a) 1/2 and 2/3
(b) 1 and 4/3
(c) 1 and 2/3
(d) 2 and 4/3

[GATE 1992: 2 Marks]

Soln. **Mean or average of any random variable is known as expected value of random variable X**

$$\boldsymbol{Mean = \mu_X = E[X] = \int_{-\infty}^{\infty} xP_X(x)dx}$$

$$\boldsymbol{= \int_{-1}^{3} x\frac{1}{4}dx = \frac{1}{4}\left[\frac{x^2}{2}\right]_1^3}$$

$$\boldsymbol{= \frac{1}{4}\left[\frac{8}{2}\right] = 1}$$

$$\boldsymbol{Variance = \sigma_x^2 = E[(X - \mu_X)] = \int_{-\infty}^{\infty} (x - \mu_X)^2 P_X(x)dx}$$

$$\boldsymbol{= \int_{-1}^{3} (x - 1)^2 \frac{dx}{4}}$$

$$\boldsymbol{= \frac{1}{4}\int_{-1}^{3} (x^2 + 1 - 2x)dx}$$

$$\boldsymbol{= \frac{1}{4}\left[\frac{x^3}{3} + x - \frac{2x^2}{2}\right]_{-1}^{3}}$$

$$\boldsymbol{= \frac{4}{3}}$$

Option (b)

6. The auto-correlation function of an energy signal has
   (a) no symmetry
   (b) conjugate symmetry
   (c) odd symmetry
   (d) even symmetry

[GATE 1996: 2 Marks]

Soln. The auto correlation is the correlation of a function with itself. If the function is real, the auto correlation function has even symmetry.

$$R_X(\tau) = R_X(-\tau)$$

The autocorrelation function has conjugate symmetry

$$R_X(\tau) = R_X^*(\tau)$$

Option (b) and (d)

7. The power spectral density of a deterministic signal is given by $[\sin(f)/f]^2$, where 'f' is frequency the autocorrelation function of this signal in the time domain is
   (a) a rectangular pulse
   (b) a delta function
   (c) a sine pulse
   (d) a triangular pulse

[GATE 1997: 2 Marks]

Soln. The Fourier transform of autocorrelation function $R_X(\tau)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, F^*(\omega) e^{j\omega\tau} d\omega$$

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, e^{j\omega\tau} d\omega$$

$$R_X(\tau) = F^{-1}[|F(\omega)|^2]$$

= Fourier inverse of power spectral density.

The auto correlation function and power spectral density make the Fourier transfer pair

$$R_X(\tau) \leftrightarrow G_X(\omega)$$

$$R_X(\tau) = F^{-1}\left[\frac{\sin f}{f}\right]^2$$

Inverse Fourier transform of square of sinc function is always a triangular signal in time domain

Option (d)

8. A probability density function is given by $P(x) = K\ exp(-x^2/2), -\infty < x < \infty$. The value of K should be

(a) $\frac{1}{\sqrt{2\pi}}$

(b) $\sqrt{\frac{2}{\pi}}$

(c) $\frac{1}{2}\sqrt{\pi}$

(d) $\frac{1}{\pi\sqrt{2}}$

[GATE 1998: 1 Mark]

Soln. Gaussian Probability density of a random variable X is given by

$$P_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left[\frac{(x-\mu)^2}{2\sigma^2}\right]}$$

When $\sigma = 1$ and $\mu = 0$ (zero mean)

$$P_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

$$Given\ P_{(x)} = ke^{\frac{-x^2}{2}}$$

$$So, k = \frac{1}{\sqrt{2\pi}}$$

Option (a)

9. The amplitude spectrum of a Gaussian pulse is

(a) uniform

(b) a sine function

(c) Gaussian

(d) an impulse function

[GATE 1998: 1 Mark]

Soln. The Fourier transform of a Gaussian signal in time domain is also Gaussian signal in the frequency domain

$$e^{-\pi t^2} \leftrightarrow e^{-\pi t^2}$$

Option (c)

10. The ACF of a rectangular pulse of duration T is
    (a) a rectangular pulse of duration T
    (b) a rectangular pulse of duration 2T
    (c) a triangular pulse of duration T
    (d) a triangular pulse of duration 2T

[GATE 1998: 1 Mark]

Soln. Autocorrelation function of a rectangular pulse of duration T is a triangular pulse of duration 2T

The autocorrelation function is an even function of τ

Option (d)

11. The probability density function of the envelope of narrow band Gaussian noise is
    (a) Poisson
    (b) Gaussian
    (c) Rayleigh
    (d) Rician

[GATE 1998: 1 Mark]

Soln. The Probability density function of the envelope of narrowband Gaussian noise is Rayleigh.

Option (c)

12. The PDF of a Gaussian random variable X is given by

$$P_X(x) = \frac{1}{3\sqrt{2\pi}} e^{\frac{-(x-4)^2}{18}}$$

The probability of the event {X=4} is

(a) $\frac{1}{2}$
(b) $\frac{1}{3\sqrt{2\pi}}$
(c) 0
(d) $\frac{1}{4}$

[GATE 2001: 1 Mark]

Soln. The probability distribution function of a Gaussian random variable X is

$$\boldsymbol{P_X(x) = \frac{1}{3\sqrt{2\pi}} e^{\frac{-(X-4)^2}{18}}}$$

The probability of a Gaussian random variable is defined for the interval and not at a point. So at X = 4, it is zero

Option (c)

13. A 1 mW video signal having a bandwidth of 100 MHz is transmitted to a receiver through a cable that has 40 dB loss. If the effective one-sided noise spectral density at the receiver is 10^{-20} Watt/Hz, then the signal-to-noise ratio at the receiver is

(a) 50 dB
(b) 30 dB
(c) 40 dB
(d) 60 dB

[GATE 2004: 2 Marks]

Soln. Signal power $= \boldsymbol{P_S = 1mw}$

Noise power $= \boldsymbol{P_N = N_0 B}$

$\boldsymbol{N_0}$ **= Noise spectral density** $= \boldsymbol{10^{-20}}$

B = bandwidth = 100 MHz

$$\boldsymbol{SNR = \frac{P_S}{P_N} = \frac{10^{-3}}{10^{-20} \times 100 \times 10^6}}$$

$$\boldsymbol{= 10^9 = 90dB}$$

Cable loss = 40 dB

SNR at receiver = 90 – 40

= 50 dB

Option (a)

14. A random variable X with uniform density in the interval 0 to 1 is quantized as follows:

If $0 \le X \le 0.3$ $\qquad x_q = 0$

If $0.3 < X \le 1,$ $\qquad x_q = 0.7$

Where x_q is the quantized value of X

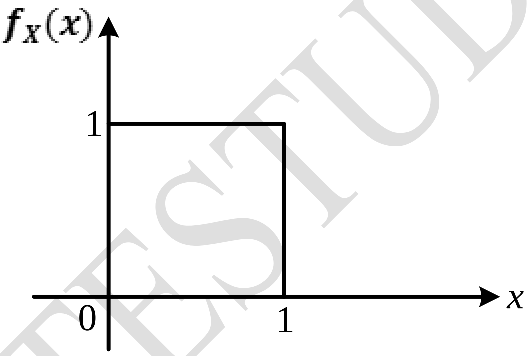
The root-mean square value of the quantization noise is

(a) 0.573 (c) 2.205

(b) 0.198 (d) 0.266

[GATE 2004: 2 Marks]

Soln.



$$\boldsymbol{0 \le X \le 0.3 \qquad x_q = 0}$$

$$\boldsymbol{0.3 \le X \le 1 \qquad x_q = 0.7}$$

$\boldsymbol{x_q}$ **is the quantized value of random variable X.**

Mean square value of the quantization noise

$$= \boldsymbol{E\left[\left(X - x_q\right)^2\right]}$$

$$= \boldsymbol{\int_0^1 \left(x - x_q\right)^2 f_X(x)dx}$$

$$= \int_{0}^{0.3} (x-0)^2 dx + \int_{0.3}^{1} (x-0.7)^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^{0.3} + \int_{0.3}^{1} \left(x^2 + 0.49 - \frac{1.4}{2}x\right) dx$$

$$\sigma^2 = 0.039$$

Root mean square value of the quantization noise

$$\sigma = \sqrt{0.039} = 0.198$$

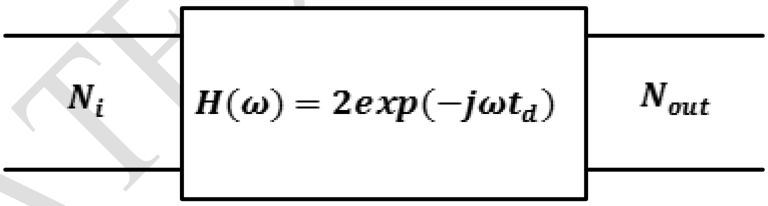
Option (b)

15. Noise with uniform power spectral density of $N_0(W/Hz)$ is passed through a filter $H(\omega) = 2exp(-j\omega t_d)$ followed by an ideal low pass filter of bandwidth B Hz. The output noise power in Watts is

(a) 2 N_0B
(b) 4 N_0B
(c) 8 N_0B
(d) 16 N_0B

[GATE 2005: 2 Marks]

Soln.



The output power spectral density of noise

$$N_{out} = |H(\omega)|^2 N_i$$

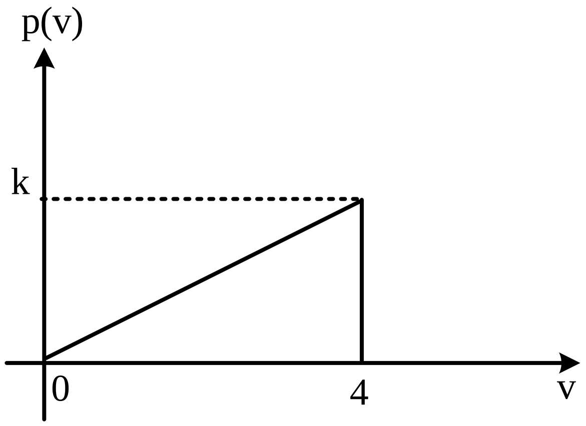
$$= 4\ N_0$$

The output noise power $P_N = 4N_0B$

The output power $P_N = 4N_0B$

Option (b)

16. An output of a communication channel is a random variable v with the probability density function as shown in the figure. The mean square value of v is



(a) 4
(b) 6
(c) 8
(d) 9

[GATE 2005: 2 Marks]

Soln. Area under the probability density function = 1

So, $\frac{1}{2} \times 4 \times k = 1$

$$k = \frac{1}{2}$$

The mean square value of the random variable X

$$E[X^2] = \int_0^4 x^2 f_X(x) dx \qquad Y = mx + C = \frac{k}{4}x$$

$$= \int_0^4 x^2 . \frac{x}{8} dx$$

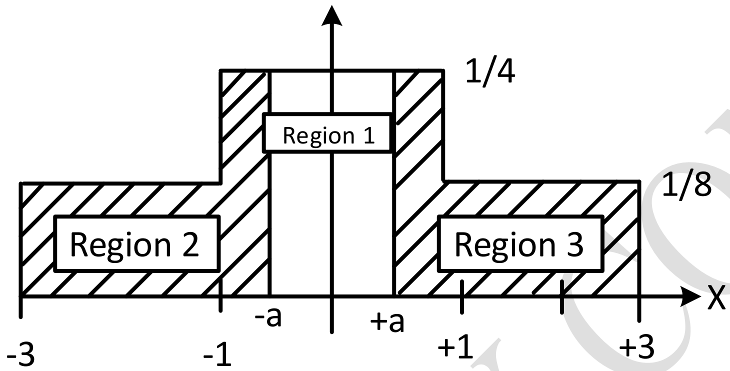
$$= \left. \frac{x^4}{8 \times 4} \right|_0^4$$

$$= \frac{4^4}{8 \times 4} = 8$$

Option (c)

Common Data for Questions 20 and 21
Asymmetric three-level midtread quantizer is to be designed assuming equiprobable occurrence of all quantization levels.

17. If the probability density function is divided into three regions as shown in the figure, the value of a in the figure is



(a) 1/3
(b) 2/3
(c) 1/2
(d) 1/4

[GATE 2005: 2 Marks]

Soln. The area under the Pdf curve must be unity. All three regions are equi -probable, thus area under each region must be $\frac{1}{3}$.

Area of region 1 $= 2a \times \frac{1}{4}$

$$\frac{2a}{4} = \frac{1}{3} \quad or\ a = \frac{2}{3}$$

Option (c)

18. The quantization noise power for the quantization region between –a and +a in the figure is

(a) $\frac{4}{81}$
(b) $\frac{1}{9}$
(c) $\frac{5}{81}$
(d) $\frac{2}{81}$

[GATE 2005: 2 Marks]

Soln. The quantization noise power for the region between –a and +a in the above figure is

$$N_q = \int_{-a}^{a} x^2 P_{(X)} dx = 2\int_{0}^{a} x^2 \frac{1}{4} dx$$

$$= \frac{2}{4}\left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{2}{4} \times \frac{a^3}{3} = \frac{a^3}{6}$$

$$a = \frac{2}{3}$$

$$So, \qquad N_q = \frac{2^3}{27 \times 6} = \frac{4}{81}$$

Option (a)

19. A zero-mean white Gaussian noise is passed through an ideal lowpass filter of bandwidth 10 KHz. The output is then uniformly sampled with sampling period $t_S = 0.03$ msec. The samples so obtained would be

(a) correlated
(b) statistically independent
(c) uncorrelated
(d) orthogonal

[GATE 2006: 2 Marks]

Soln. White noise contains all frequency components, but the phase relationship of the components is random. When white noise is sampled, the samples are uncorrelated. If white noise is Gaussian, the samples are statistically independent

Option (b)