Probability and Random Processes (Part – II)

1. If the variance σ_x^2 $of\ d(n) = x(n) - x(n-1)$ is one-tenth the variance σ_x^2 of a stationary zero-mean discrete-time signal $x(n)$, then the normalized autocorrelation function $R_{xx}(k)/\sigma_x^2$ at k = 1 is

   (a) 0.95 (c) 0.10

   (b) 0.90 (d) 0.05

[GATE 2002: 2 Marks]

Soln. The variance $\boldsymbol{\sigma_X^2 = E[(X - \mu_X)^2]}$

Where $\boldsymbol{\mu_X(mean\ value) = 0}$

$$\boldsymbol{\sigma_d^2 = E[\{X(n) - X(n-1)\}^2]}$$

$$\boldsymbol{\sigma_d^2 = E[X(n)]^2 + E[X(n-1)]^2 - 2E[X(n)X(n-1)]}$$

$$\boldsymbol{\frac{\sigma_X^2}{10} = \sigma_X^2 + \sigma_X^2 - 2R_{XX}(1)}$$

$$\boldsymbol{\sigma_X^2 = 20\sigma_X^2 - 20R_{XX}(1)}$$

$$\boldsymbol{\frac{R_{XX}}{\sigma_X^2} = \frac{19}{20} = 0.95}$$

Option (a)

2. Let Y and Z be the random variables obtained by sampling $X(t)$ at t = 2 and t = 4 respectively. Let W = Y – Z. The variance of W is

   (a) 13.36 (c) 2.64

   (b) 9.36 (d) 8.00

[GATE 2003: 2 Marks]

Soln. $W = Y - Z$ $\quad\quad$ $Given\ R_{XX(\tau)} = 4\left(e^{-0.2|\tau|} + 1\right)$

$$\text{Variance}[W] = E[Y - Z]^2$$

$$\sigma_W^2 = E[Y^2] + E[Z^2] - 2E[YZ]$$

Y and Z are samples of X(t) at t = 2 and t = 4

$$E[Y^2] = E[X^2(2)] = R_{XX(0)}$$

$$= 4\left[e^{-.2|0|} + 1\right] = 8$$

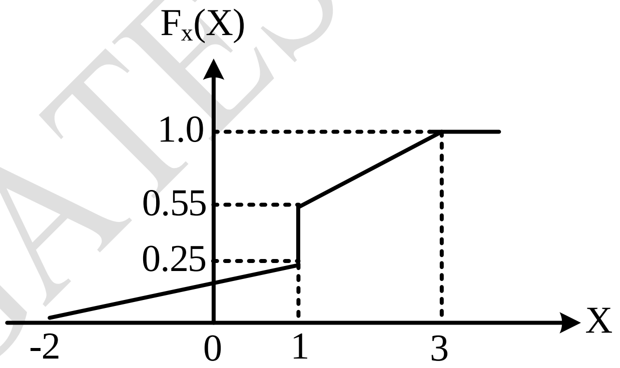
$$E[Z^2] = E[X^2(4)] = 4\left[e^{-0.2|0|} + 1\right] = 8$$

$$E[YZ] = R_{XX(2)} = 4\left[e^{-0.2(4-2)} + 1\right] = 6.68$$

$$\sigma_W^2 = 8 + 8 - 2 \times 6.68 = 2.64$$

Option (c)

3. The distribution function $F_X(x)$ of a random variable X is shown in the figure. The probability that X = 1 is



(a) Zero
(b) 0.25
(c) 0.55
(d) 0.30

[GATE 2004: 1 Mark]

Soln. The probability that $X = 1 = F_X(x = 1^+) - F_X(x = 1^-)$

$$P(x = 1) = 0.55 - 0.25 = 0.30$$

Option (d)

4. If E denotes expectation, the variance of a random variable X is given by

(a) $E[X^2] - E^2[X]$ (c) $E[X^2]$

(b) $E[X^2] + E^2[X]$ (d) $E^2[X]$

[GATE 2007: 1 Mark]

Soln. The variance of random variable X

$$\boldsymbol{\sigma_X^2 = E[(X - \mu_X)^2]}$$

Where $\boldsymbol{\mu_X}$ is the mean value = $\boldsymbol{E[X]}$

$$\boldsymbol{\sigma_X^2 = E[X^2] + E[\mu_X]^2 - 2\,\mu_X E[X]}$$

$$\boldsymbol{= E[X^2] + \mu_X^2 - 2\,\mu_X\mu_X}$$

$$\boldsymbol{= E[X^2] - \mu_X^2}$$

= mean square value – square of mean value

Option (a)

5. If $R(\tau)$ is the auto-correlation function of a real, wide-sense stationary random process, then which of the following is NOT true?

(a) $R(\tau) = R(-\tau)$

(b) $|R(\tau)| \leq R(0)$

(c) $R(\tau) = -R(-\tau)$

(d) The mean square value of the process is R(0)

[GATE 2007: 1 Mark]

Soln. If all the statistical properties of a random process are independent of time, it is known as stationary process.

The autocorrelation function is the measure of similarity of a function with it's delayed replica.

$$\boldsymbol{R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t-\tau)\, f^*(t)\, dt}$$

$$for \quad \tau = 0, \; R(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, f^*(t) dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2\, dt$$

R(0) is the average power P of the signal.

$R(\tau) = R^*(-\tau)$ ***exhibits conjugate symmety***

$R(\tau) = R(-\tau)$ ***for real function***

$R(0) \geq R(\tau)$ ***for all*** τ

$R(\tau) = -R(-\tau)$ **is not true (since it has even symmetry)**

Option (c)

6. If S(f) is the power spectral density of a real, wide-sense stationary random process, then which of the following is ALWAYS true?
   (a) $S(0) \geq S(f)$
   (b) $S(f) \geq 0$
   (c) $S(-f) = -S(f)$
   (d) $\int_{-\infty}^{\infty} S(f)df = 0$

[GATE 2007: 1 Mark]

Soln. Power spectral density is always positive

$S(f) \geq 0$

Option (b)

7. $P_X(x) = M \exp(-2|x|) + N\, exp(-3|x|)$ is the probability density function for the real random variable X over the entire X axis M and N are both positive real numbers. The equation relating M and N is
   (a) $M + \frac{2}{3}N = 1$
   (b) $2M + \frac{1}{3}N = 1$
   (c) $M + N = 1$
   (d) $M + N = 3$

[GATE 2008: 2 Marks]

Soln.

$$\int_{-\infty}^{\infty} \boldsymbol{P_X(x)\, dx = 1}$$

$$\int_{-\infty}^{\infty} \boldsymbol{(M.e^{-2x} + N.e^{-3x})\, dx = 1}$$

$$\int_{0}^{\infty} \boldsymbol{(M.e^{-2x} + N.e^{-3x})\, dx = \frac{1}{2}}$$

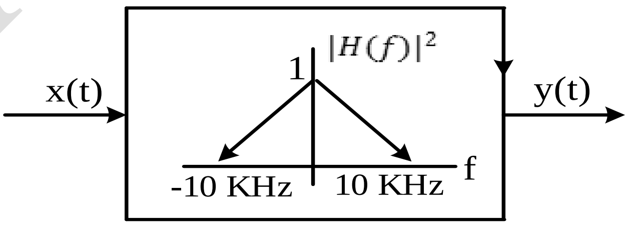
$$\boldsymbol{\frac{M.e^{-2x}}{-2}\Bigg|_0^{\infty} + \frac{N.e^{-3x}}{-3}\Bigg|_0^{\infty} = \frac{1}{2}}$$

$$\boldsymbol{\frac{M}{2} + \frac{N}{3} = \frac{1}{2}}$$

$$\boldsymbol{or, \qquad M + \frac{2N}{3} = 1}$$

Option (a)

8. A white noise process X(t) with two-sided power spectral density $1 \times 10^{-10}\ W/Hz$ is input to a filter whose magnitude squared response is shown below.



The power of the output process y (t) is given by

(a) $5 \times 10^{-7} W$

(b) $1 \times 10^{-6} W$

(c) $2 \times 10^{-6} W$

(d) $1 \times 10^{-5} W$

[GATE 2009: 1 Mark]

Soln. **Power spectral density of white noise at the input of a filter** $= G_i(f)$

$$G_i(f) = 1 \times 10^{-10} (W/Hz)$$

PSD at the output of a filter

$$G_0(f) = |H(f)|^2 G_i(f)$$

$$= \frac{1}{2}(2 \times 10 \times 10^3 \times 1) \times 10^{-10}$$

$$= 10^{-6} W$$

Option (b)

9. Consider two independent random variables X and Y with identical distributions. The variables X and Y take value 0,1 and 2 with probabilities $\frac{1}{2}$, $\frac{1}{4}$ *and* $\frac{1}{4}$ respectively. What is the conditional probability $(X + Y = 2 | X - Y = 0)$?

(a) 0
(b) 1/16
(c) 1/6
(d) 1

[GATE 2009: 2 Marks]

Soln.

$$P(X = 0) = P(Y = 0) = \frac{1}{2}$$

$$P(X = 1) = P(Y = 1) = \frac{1}{4}$$

$$P(X = 2) = P(Y = 2) = \frac{1}{4}$$

$$P(X - Y = 0) = P(X = 0, Y = 0) + P(X = 1, Y = 1)$$

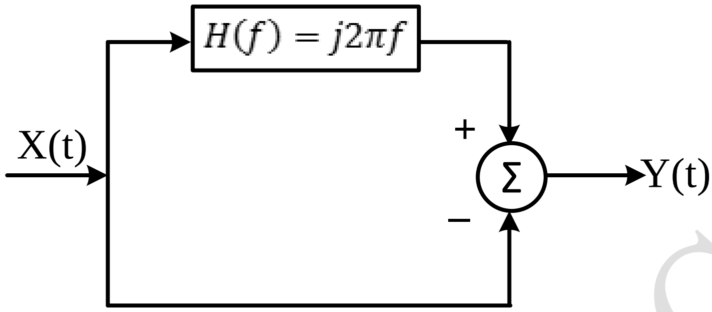
$$+ P(X = 2, Y = 2) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{6}{16}$$

$$P(X + Y = 2) = P(X = 1, Y = 1) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

$$P(X+Y=2\,|_{X-Y=0}) = \frac{1}{16} \div \frac{6}{16} = 1/6$$

Option (c)

10.X (t) is a stationary random process with autocorrelation function $R_X(\tau) = exp(-\pi\tau^2)$ this process is passed through the system below. The power spectral density of the output process Y(t) is



(a) $(4\pi^2 f^2 + 1)\ exp(-\pi f^2)$
(b) $(4\pi^2 f^2 - 1)\ exp(-\pi f^2)$
(c) $(4\pi^2 f^2 + 1)\ exp(-\pi f)$
(d) $(4\pi^2 f^2 - 1)\ exp(-\pi f)$

[GATE 2011: 2 Marks]

Soln.

$$Y(f) = j2\pi f\, X(f) - X(f)$$

$$PSD\ \ S_Y(f) = \left|(j2\pi f - 1)^2\right| S_X(f)$$

$$S_X(f) = FT\{R_X(\tau)\}$$

$$= FT(e^{-\pi\tau 2})$$

$$= e^{-\pi f^2}$$

$$S_Y(f) = (4\pi^2 f^2 + 1)e^{-\pi f^2}$$

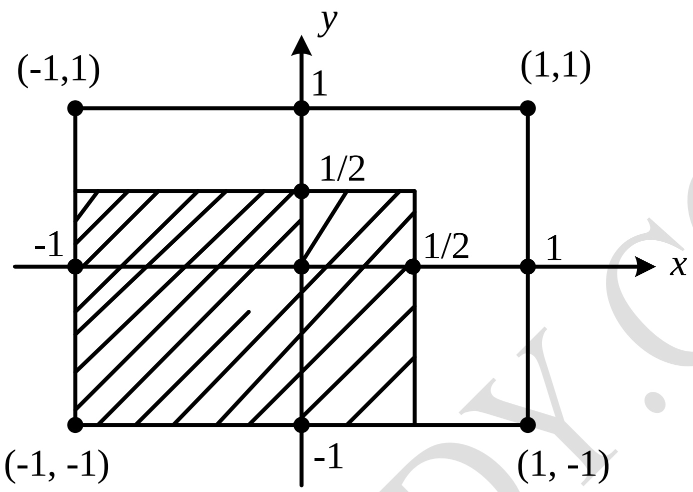
Option (a)

11. Two independent random variables X and Y are uniformly distributed in the interval $[-1,1]$. The probability that max $[X, Y]$ is less than 1/2 is

(a) 3/4
(b) 9/16
(c) 1/4
(d) 2/3

[GATE 2012: 1 Mark]

Soln.



$$-1 \leq X \leq 1 \quad and \quad -1 \leq Y \leq 1$$

The region in which maximum of $[X, Y]$ is less than 1/2 is shown as shaded region inside the rectangle.

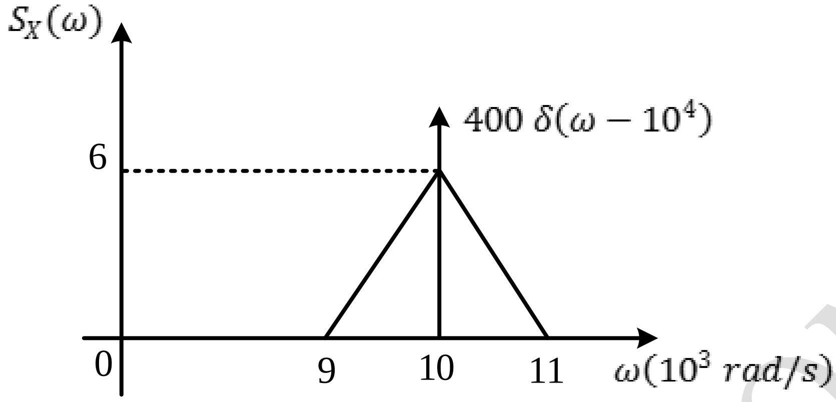
$$P\left[\max(X, Y) < \frac{1}{2}\right] = \frac{Area\ of\ shaded\ region}{Area\ of\ entire\ region}$$

$$= \frac{\frac{3}{2} \times \frac{3}{2}}{2 \times 2} = \frac{9}{4 \times 4}$$

$$= \frac{9}{16}$$

Option (b)

12. A power spectral density of a real process X (t) for positive frequencies is shown below. The values of $[E[X^2(t)]$ and $|E[X(t)]|]$ respectively are



(a) $6000/\pi, 0$
(b) $6400/\pi, 0$
(c) $6400/\pi, 20/(\pi\sqrt{2})$
(d) $6000/\pi, 20/(\pi\sqrt{2})$

[GATE 2012: 1 Mark]

Soln. The mean square value of a stationary process equals the total area under the graph of power spectral density

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)df$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega$$

$$= \frac{2}{2\pi}\int_{0}^{\infty} S_X(\omega)\, d\omega$$

$$= \frac{1}{\pi}[area\ under\ the\ triangle + integration\ under\ delta\ function]$$

$$= \frac{1}{\pi}\left[2\left(\frac{1}{2} \times 1 \times 6 \times 10^3\right) + 400\right]$$

$$= \frac{6400}{\pi}$$

$|E[X(t)]|$ is the absolute value of mean of signal $X(t)$ which is also equal to value of $X(\omega)$ at $\omega = 0$

From PSD

$$S_X(\omega)|_{\omega=0} = 0$$

$$|X(\omega)|^2 = 0$$

$$|X(\omega)| = 0$$

Option (b)

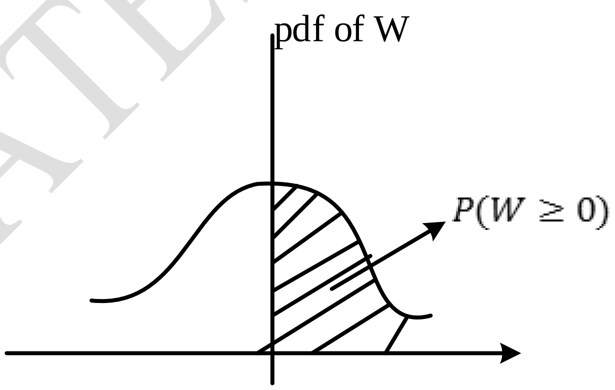
13. Let U and V be two independent zero mean Gaussian random variables of variances $\frac{1}{4}$ and $\frac{1}{9}$ respectively. The probability $P(3V \geq 2U)$ is

(a) 4/9 (c) 2/3

(b) 1/2 (d) 5/9

[GATE 2013: 2 Marks]

Soln.



$$P(3V - 2U) = P(3V - 2U \geq 0)$$

$$= P(W \geq 0)$$

$$W = 3V - 2U$$

W is the Gaussian Variable with zero mean having pdf curve as shown below

$\boldsymbol{P(W \geq 0) = \frac{1}{2}(area\ under\ the\ curve\ from\ 0\ to\ \infty)}$

Option (b)

14. Let $X_1, X_2, and\ X_3$ be independent and identically distributed random variables with the uniform distribution on $[0,1]$. The probability $P\{X_1\ is\ the\ largest\}$ is ________

[GATE 2014: 1 Mark]

Soln. Probability $\boldsymbol{P[X_1] = P[X_2] = P[X_3]}$

$$\boldsymbol{P_1 + P_2 + P_3 = 1}$$

$$\boldsymbol{P(X_1) + P(X_2) + P(X_3) = 1}$$

$$\boldsymbol{3P(X_1) = 1}$$

$$\boldsymbol{P(X_1) = \frac{1}{3}}$$

15. Let X be a real-valued random variable with $E[X]\ and\ E[X^2]$ denoting the mean values of X and X^2, respectively. The relation which always holds

(a) $(E[X])^2 > E[X^2]$
(b) $E[X^2] \geq (E[X])^2$
(c) $E[X^2] = (E[X])^2$
(d) $E[X]^2 > (E[X])^2$

[GATE 2014: 2 Marks]

Soln. variance $\boldsymbol{\sigma_X{}^2 = E[X^2] - m_X{}^2}$

$$\boldsymbol{\overline{X^2} - m_X{}^2}$$

= mean square value – square of mean value

$$\boldsymbol{\sigma_X{}^2 = E[X^2] - [E(X)]^2}$$

Variance is always positive so $\boldsymbol{E[X^2] \geq [E(X)^2]}$

And can be zero

Option (b)

16.Consider a random process $X(t) = \sqrt{2}\sin(2\pi t + \phi)$, where the random phase ϕ is uniformly distributed in the interval$[0,2\pi]$. The autocorrelation $E[X(t_1)X(t_2)]$ is

(a) $\cos[2\pi(t_1 + t_2)]$

(b) $\sin[2\pi(t_1 - t_2)]$

(c) $\sin[2\pi(t_1 + t_2)]$

(d) $\cos[2\pi(t_1 - t_2)]$

[GATE 2014: 2 Marks]

Soln. $\boldsymbol{E[X(t_1)\,X(t_2)] = E[A\sin(2\pi t_1 + \phi) \times A\sin(2\pi t_2 + \phi)]}$

$$= \frac{A^2}{2}E[\cos 2\pi(t_1 - t_2) - \cos 2\pi(t_1 + t_2 + 2\phi)]$$

$$= \frac{A^2}{2}\cos 2\pi(t_1 - t_2)$$

$\boldsymbol{E[\cos 2\pi(t_1 + t_2 + 2\phi)] = 0}$

Option (d)

17. Let X be a random variable which is uniformly chosen from the set of positive odd numbers less than 100. The expectation, E[X] is

[GATE 2014: 1 Mark]

Soln.

$$E[X] = \frac{1 + 3 + 5 + - - - - - (2n - 1)}{50}$$

Where n = 50

$$= \frac{n^2}{50} = 50$$

18. The input to a 1-bit quantizer is a random variable X with pdf $f_X(x) = 2e^{-2x}\ for\ x \geq 0$ and$f_X(x) = 0\ for\ x < 0$. For outputs to be of equal probability, the quantizer threshold should be______

[GATE 2014: 2 Marks]

Soln. **The input to a 1-bit quantizer is a random variable X with pdf**

$f_X(x) = 2e^{-2x}\ for\ x \geq 0$ **And** $f_X(x) = 0\ for\ x < 0$ **let** V_{thr} **be the quantizer threshold**

$$\int_{-\infty}^{V_{thr}} 2e^{-2x}\,dx = \int_{V_{thr}}^{\infty} 2e^{-2x}\,dx$$

$$= \int_{0}^{V_{thr}} 2e^{-2x}\,dx = \int_{V_{thr}}^{\infty} 2e^{-2x}\,dx \quad f_X(x) = 0\ for\ x < 0$$

$$\left.\frac{2e^{-2x}}{-2}\right|_0^{V_{thr}} = \left.\frac{2e^{-2x}}{-2}\right|_{V_{thr}}^{\infty}$$

$$(-e^{-2V_{thr}} + e^{-0}) = -(0 - e^{-2V_{thr}})$$

$$e^{-2V_{thr}} = \frac{1}{2}$$

$$-2V_{thr} = ln\left(\frac{1}{2}\right) = (-0.693)$$

$$V_{thr} = \frac{0.693}{2}$$

= 0.346