Information Theory and Coding

1. The capacity of a band-limited additive white Gaussian (AWGN) channel is given by
   $C = W log_2 \left(1 + \frac{P}{\sigma^2 W}\right)$ bits per second(bps), where W is the channel bandwidth, P is the average power received and σ^2 is the one-sided power spectral density of the AWGN.
   For a fixed $\frac{P}{\sigma^2} = 1000$, the channel capacity (in kbps) with infinite bandwidth ($W \to \infty$) is approximately
   (a) 1.44
   (b) 1.08
   (c) 0.72
   (d) 0.36

[GATE 2014: 1 Mark]

Soln.

$$C = W\ log_2 \left(1 + \frac{P}{\sigma^2 W}\right)$$

$$= \frac{P}{\sigma^2} \times \frac{\sigma^2}{P}\ W\ log_2 \left(1 + \frac{P}{\sigma^2 W}\right)$$

$$= \frac{P}{\sigma^2} \times \frac{\sigma^2}{P}\ W\ log_2 \left(1 + \frac{1}{\sigma^2 W/_P}\right)$$

$$= \frac{P}{\sigma^2} \lim_{\to\infty} \left[x\ log_2 \left(1 + \frac{1}{x}\right)\right] \qquad where\ x = \frac{\sigma^2 W}{P}$$

$$= \frac{P}{\sigma^2}\ log_2 e$$

$$= 1.44 \times \frac{P}{\sigma^2}$$

$$= 1.44 \times 1000 = 1.44\ kbps$$

Option (a)

2. A fair is tossed repeatedly until a ‘Head’ appears for the first time. Let L be the number of tosses to get this first ‘Head’. The entropy H(L) in bits is _________

[GATE 2014: 2 Marks]

Soln. **If 1 toss is required to get first head, then probability** $= \frac{1}{2}$

If 2 tosses are required to get first head then $P_2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

If 3 tosses are required to get first head then $P_3 = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

Entropy

$$H = \sum_{i=1}^{n} P_i log_2 \frac{1}{P_i}$$

$$= \frac{1}{2} log_2 2 + \frac{1}{4} log_2 4 + \frac{1}{8} log_2 8 + \frac{1}{16} log_2 16$$

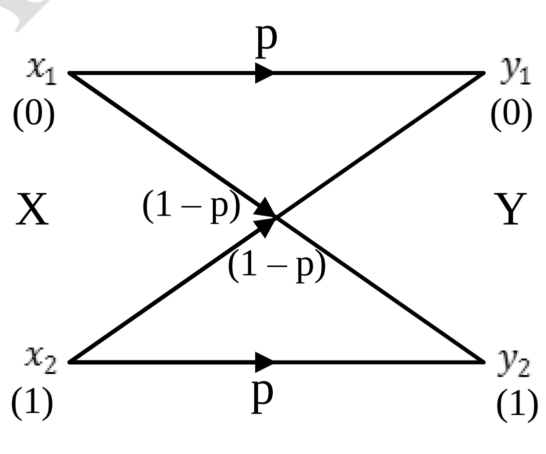
$$= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4}$$

$$\cong 2$$

3. The capacity of a Binary Symmetric Channel (BSC) with cross-over probability 0.5 is ________

[GATE 2014: 1 Mark]

Soln.



Given cross over probability of 0.5

$\boldsymbol{P(x_1) = {}^1/_2}$

$\boldsymbol{P(x_2) = {}^1/_2}$

Channel capacity for BSC

$$(\boldsymbol{C}) = \boldsymbol{log_2 n} - \left[-\sum_{j=1}^{2} \boldsymbol{p}\left({}^{y_k}/_{x_j}\right) \log \boldsymbol{p}\left({}^{y_k}/_{x_j}\right)\right]$$

$$\boldsymbol{log_2 2 + p \log p + (1-p)\log(1-p)}$$

$$= \boldsymbol{1 + \frac{1}{2} log_2(1/2) + \frac{1}{2} log_2(1/2)}$$

$$= \boldsymbol{1 - \frac{1}{2} - \frac{1}{2} = 0}$$

$\boldsymbol{C = 0}$

$\boldsymbol{Capacity = 0}$

4. In a digital communication system, transmission of successive bits through a noisy channel are assumed to be independent events with error probability p. The probability of at most one error in the transmission of an 8-bit sequence is
(a) $7(1-p)/+p/8$
(b) $(1-p)^8 + 8P(1-p)^7$
(c) $(1-p)^8 + (1-p)^7$
(d) $(1-p)^8 + p(1-p)^7$

[GATE 1988: 2 Marks]

Soln. Getting almost one error be success

Probability of at most one error = p

Say, success

Failure = 1 – p

P (X = at most 1 error)

$$= \boldsymbol{P(X=0) + P(X=1)}$$

Note that probability that event A occurs r times is given by bionomical probability man function defined as

$\boldsymbol{P(X=r)}$ ^nC_r $\boldsymbol{p^r(1-p)^{n-r}}$

$= {}^8C_0\ (p)^0(1-p)^{8-0} + {}^8C_1\ (p)^1(1-p)^{8-1}$

$= (1-p)^8 + 8p\ (1-p)^7$

Option (b)

5. Consider a Binary Symmetric Channel (BSC) with probability of error being p. To transmit a bit say 1, we transmit a sequence of three sequence to represent 1 if at least two bits bit will be represent in error is

(a) $p^3 + 3p^2(1-p)$

(b) p^3

(c) $(1-p)^3$

(d) $p^3 + p^2(1-p)$

[GATE 2008: 2 Marks]

Soln. $P(0/_1) = P(1/_0) = p$

$P(1/_1) = P(0/_0) = 1 - P$

Reception with error means getting at the most one 1.

P(reception with error)

$= P(X=0) + P(X=1)$

Using the relation of Binomial probability man function

$P(X=r) = {}^nC_r\ p^r(1-p)^{n-r}$

For r = 0, 1, 2, -------- n

$= {}^3C_0\ (1-p)^0p^3 + {}^3C_1\ (1-p)^1p^2$

$= p^3 + 3p^2(1-p)$

Option (a)

6. During transmission over a certain binary communication channel, bit errors occur independently with probability p. The probability of at most one bit in error in a block of n bits is given by

(a) p^n

(b) $1-p^n$

(c) $np(1-p)^{n-1} + (1-p)^n$

(d) $1-(1-p)^n$

[GATE 2007: 2 Marks]

Soln. **Probability of at most one bit is error**

P = P (non error)+P(one bit error)

Using the relation of Binomial probability man function

$= {}^{n}C_0\,(p)^0(1-p)^n + {}^{n}C_1\,(p)^1(1-p)^{n-1}$

$= (1-p)^n + np(1-p)^{n-1}$

Note, $^{n}C_0 = 1$ **and** $^{n}C_1 = n$

Option (c)

7. Let U and V be two independent and independent and identically distributed random variables such that $P(U=+1) = P(U=-1) = \frac{1}{2}$. The entropy H(U+V) in bits is

(a) 3/4
(b) 1
(c) 3/2
(d) $\log_2 3$

[GATE 2013: 2 Marks]

Soln. **U and V are two independent and identically distributed random variables**

$P(U=+1) = P(U=-1) = \frac{1}{2}$

$P(V=+1) = P(V=-1) = \frac{1}{2}$

So, random variables U and V can have following values

$U = +1, -1; \quad V = +1, -1$

$$U+V\begin{cases} -2 & When\ U=V=-1 \\ 0 & when\ U=1,\ V=-1 \quad or\ U=-1,\ V=1, \\ 2 & when\ U=V=1 \end{cases}$$

$U+V=-2 \qquad P(U+V)=-2=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

$U+V=0 \qquad P(U+V)=0=\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{1}{2}$

$U+V=2 \qquad P(U+V)=2=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

Entropy of $(U+V) = H(U+V)$

$$= \sum P(U+V)\, log_2 \frac{1}{P(U+V)}$$

$$= \frac{1}{4} log_2 4 + \frac{1}{2} log_2 2 + \frac{1}{4} log_2 4$$

$$= \frac{2}{4} + \frac{1}{2} + \frac{2}{4} = \frac{3}{2}$$

Option (c)

8. A source alphabet consists of N symbols with the probability of the first two symbols being the same. A source encoder increases the probability of the first symbol by a small amount e. After encoding, the entropy of the source
(a) increases
(b) remains the same
(c) increases only if N = 2
(d) decreases

[GATE 2012: 1 Mark]

Soln. Entropy is maximum, when symbols are equally probable, when probability changes from equal to non-equal, entropy decreases

Option (d)

9. A communication channel with AWGN operating at a signal at a signal to noise ratio SNR >>1 and bandwidth B has capacity C_1. If the SNR is doubled keeping B constant, the resulting capacity C_2 is given by
(a) $C_2 \approx 2C_1$
(b) $C_2 \approx C_1 + B$
(c) $C_2 \approx C_1 + 2B$
(d) $C_2 \approx C_1 + 0.3B$

[GATE 2009: 2 Marks]

Soln. When SNR >>1, channel capacity C

$$C_1 = B\, log_2\left(1 + \frac{S}{N}\right)$$

$$C_1 \approx B\, log_2\left(\frac{S}{N}\right)$$

When SRN is doubled

$$C \approx B\, log_2\left(\frac{2S}{N}\right) = B\, log_2 2 + B\, log_2\left(\frac{S}{N}\right)$$

$$C = B\ log_2\left(\frac{S}{N}\right) + B$$

$$= C_1 + B$$

Option (b)

10. A memoryless source emits n symbols each with a probability p. The entropy of the source as a function of n

(a) increases
(b) decreases as log n
(c) increases as n
(d) increases as n log n

[GATE 2008: 2 Marks]

Soln. Entropy H(m) for the memoryless source

$$H(m) = -\sum_{i=1}^{n} P_i\ log_2\ P_i \qquad bits$$

P_i = Probability of individual symbol

$$P_1 = P_2 = - - - - - - - P_n = \frac{1}{n}$$

$$H(m) = -\sum_{i=1}^{n} \frac{1}{n} log_2 \frac{1}{n}$$

$$= \frac{1}{n} log_2 n$$

Entropy H(m) increases as a function of $log_2 n$

Option (a)

11. A source generates three symbols with probability 0.25, 0.25, 0.50 at a rate of 3000 symbols per second. Assuming independent generation of symbols, the most efficient source encoder would have average bit rate of

(a) 6000 bits/sec
(b) 4500 bits/sec
(c) 3000 bits/sec
(d) 1500 bits/sec

[GATE 2006: 2 Marks]

Soln. Three symbols with probability of 0.25, 0.25 and 0.50 at the rate of 3000 symbols per second.

$$\boldsymbol{Entropy\ H = 0.25\ log_2 \frac{1}{0.25} + 0.25\ log_2 \frac{1}{0.25} + 0.5\ log_2 \frac{1}{0.5}}$$

$$\boldsymbol{= 0.25 \times 2 + 0.25 \times 2 + 0.5}$$

$$\boldsymbol{= 1.5}$$

Rate of information R = r.H

R = 3000 symbol/sec

$$\boldsymbol{R = 3000 \times 1.5}$$

$$\boldsymbol{= 4500\ bits/sec}$$

Option (b)

12. An image uses 512×512 picture elements. Each of the picture elements can take any of the 8 distinguishable intensity levels. The maximum entropy in the above image will be

(a) 2097152 bits
(b) 786432 bits
(c) 648 bits
(d) 144 bits

[GATE 1990: 2 Marks]

Soln. For 8 distinguishable intensity levels

$$\boldsymbol{n = log_2 L}$$

$$\boldsymbol{n = log_2 8 = 3}$$

Maximum entropy = $\boldsymbol{512 \times 512 \times n}$

$$\boldsymbol{= 512 \times 512 \times 3}$$

$$\boldsymbol{= 786432}$$

13. A source produces 4 symbols with probability $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\ and\ \frac{1}{8}$. For this source, a practical coding scheme has an average codeword length of 2 bits/symbols. The efficiency the code is

(a) 1
(b) 7/8
(c) 1/2
(d) 1/4

[GATE 1989: 2 Marks]

Soln. Four symbol with probability $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ *and* $\frac{1}{8}$

$$Entoropy = H = -\sum_{i=1}^{n} P_i log_2(P_i)$$

$$H = -\left[\frac{1}{2} log_2\left(\frac{1}{2}\right) + \frac{1}{4} log_2\left(\frac{1}{4}\right) + \frac{1}{8} log_2\left(\frac{1}{8}\right) + \frac{1}{8} log_2\left(\frac{1}{8}\right)\right]$$

$$= \frac{1}{2} + \frac{1}{4} \times 2 + \frac{1}{8} \times 3 + \frac{1}{8} \times 3$$

$$= 1 + \frac{3}{4} = \frac{7}{4}$$

Code efficiency $\frac{H}{L}$

$$= \frac{7}{4 \times 2}$$

$$= \frac{7}{8}$$

Option (b)