Z – Transform (Part - II)

1. The Z – Transform of the following real exponential sequence
   $x(nT) = a^n, nT \geq 0$
   $\quad\quad = 0, nT < 0,\ a > 0$
   (a) $\frac{1}{1-z^{-1}};\ |z| > 1$
   (b) $\frac{1}{1-az^{-1}};\ |z| > a$
   (c) $1\ for\ all\ z$
   (d) $\frac{1}{1-az^{-1}};\ |z| < a$

[GATE 1990: 2 Marks]

Soln. The given sequence can be written as

$x(n)a^n(n)$ where a > 0

Z – Transform of given sequence is

$$X(z) = \frac{z}{z-a};\ |z| > |a| = \frac{1}{1-az^{-1}}, |z| > a$$

Option (b)

2. A linear discrete-time system has the characteristics equation,
   $z^3 - 0.81\ z = 0$. The system
   (a) is stable
   (b) is marginally stable
   (c) is unstable
   (d) stability cannot be assessed from the given information

[GATE 1992: Marks]

Soln. Characteristic equation is given

$$z^3 - 0.81z = 0$$

$$z(z^2 - 0.81) = 0$$

$$z(z - 0.9)(z + 0.9) = 0$$

So, poles are

$$z = 0, \quad 0.9 \quad and - 0.9$$

Note that all three poles are inside the unit circle, so the system is stable

Option (a)

3. The z – transform of a signal is given by

$$C(z) = \frac{1}{4}\frac{z^{-1}(1-z^{-4})}{(1-z^{-1})^2}\ .\ Its\ final\ value\ is$$

(a) 1/4
(b) zero
(c) 1.0
(d) infinity

[GATE 1999: 2 Marks]

Soln. Final value theorem for Z – Transform is

$$\lim_{N\to\infty} x[N] = \lim_{z\to 1}(1-z^{-1})\,X(z)$$

$$= \lim_{z\to 1}\frac{1}{4}.\frac{(1-z^{-1})\,z^{-1}\,(1-z^{-4})}{(1-z^{-1})^2}$$

$$= \lim_{z\to 1}\ \frac{1}{4}\ \frac{z^{-1}(1-z^{-4})}{(1-z^{-1})}$$

$$= \lim_{z\to 1}\ \frac{1}{4}.\frac{1}{z}.\frac{(z^4-1)/z^4}{(z-1)/z}$$

$$= \lim_{z\to 1}\ \frac{1}{4}\ \frac{(z^2-1)(z^2+1)}{z^4(z-1)}$$

$$= \lim_{z\to 1}\ \frac{1}{4}\ z^2\,\frac{(z+1)(z-1)(z^2+1)}{(z-1)}$$

$$= \lim_{z\to 1}\ \frac{1}{4}\ z^2\,(z+1)(z^2+1) = \frac{1}{4}\ .1^2\ .2\ .2 = 1$$

Option (a)

4. If the impulse response of a discrete-time system is $h[n] = -5^n u[-n-1]$. Then the system function H(z) is equal to

(a) $\frac{-z}{z-5}$ and the system is stable
(b) $\frac{z}{z-5}$ and the system is stable
(c) $\frac{-z}{z-5}$ and the system is unstable
(d) $\frac{z}{z-5}$ and the system is unstable

[GATE 2002: 2 Marks]

Soln. Impulse response $h(n) = -5^n u[-n-1]$

Above response is having left handed sequence whose z – transform has standard form

$$-a^n u(n)(-n-1) \xleftrightarrow{\quad z \quad} \frac{1}{(1-a\,z^{-1})}\ ;\ \ ROC : |z| < a$$

Thus

$$-5^n u(n)(-n-1) \longleftrightarrow \frac{1}{(1-5\,z^{-1})}\ ;\ \ ROC : |z| < 5$$

or,

$$-5^n u(n)(-n-1) \longleftrightarrow \frac{z}{z-5}\ ;\ \ ROC : |z| < 5$$

since ROC is $|z| < 5$ so it includes unit circle, system is stable

Option (b)

5. A causal LTI system is described by the difference equation
$2y[n] = \alpha y[n-2] - 2x[n] + \beta x[n-1]$
The system is stable only if
(a) $|\alpha| = 2, \quad |\beta| < 2$
(b) $|\alpha| > 2, \quad |\beta| > 2$
(c) $|\alpha| < 2, \quad any\ value\ of\ \beta$
(d) $|\beta| < 2, \quad any\ value\ of\ \alpha$

[GATE 2004: 2 Marks]

Soln. Casual LTI system is described by the difference equation

$$2\,y[n] = \alpha\,y[n-2] - 2\,x[n] + \beta\,x[n-1]$$

Taking z – transform

$$2\,Y[z] = \alpha\,Y(z)z^{-2} - 2\,X(z) + \beta\,X(z)z^{-1}$$

$$\frac{Y(z)}{X(z)} = \frac{(\beta\,z^{-1}-2)}{(2-\alpha\,z^{-2})}$$

Or, $H(z) = \frac{z(\beta/2 - z)}{(z^2 - \alpha/2)}$

It has poles at $\pm\sqrt{\alpha/_2}$ and zero at 0 and $^{\beta}/_{2}$

For stable system poles must lie inside the unit circle of z – plane.

But 0 can lie any where in plane. Thus β can be of any value

Option (c)

6. The z – transform $X[z]$ of a sequence $x[n]$ is given by $X[z] = \frac{0.5}{1-2z^{-1}}$. It is given that the Region of convergence of $X[z]$ includes the unit circle. The value of $x[0]$ is

(a) – 0.5
(b) 0
(c) 0.25
(d) 0.5

[GATE 2007: 2 Marks]

Soln. Given

$X(z)$ of a sequence $x[n]$

$$X(z) = \frac{0.5}{1-2\,z^{-1}}$$

Above transform is for left handed sequence with

$$\text{ROC}: \ |z| < 2$$

Corresponding sequence is

$$x(n) = -(0.5)2^{-n}u(-n-1)$$

So, $x(0) = 0$

If, the value of $x(0)$ is determined by initial value theorem then it will not be correct, since the sequence $x(n)$ is defined for $n < 0$

Option (b)

7. A system with transfer function H(z) has impulse response h(n) defined as h(2) = 1, h(3) = -1 and h(k) = 0 otherwise. Consider the following statements.

S_1 : H(z) is low-pass filter

S_2 : H(z) is an FIR filter

Which of the following is correct?

(a) Only S_2 is true

(b) Both S_1 is and S_2 are false

(c) Both S_1 is and S_2 are true, and S_2 is a reason for S_1

(d) Both S_1 is and S_2 are true, and S_2 is not a reason for S_1

[GATE 2009: 2 Marks]

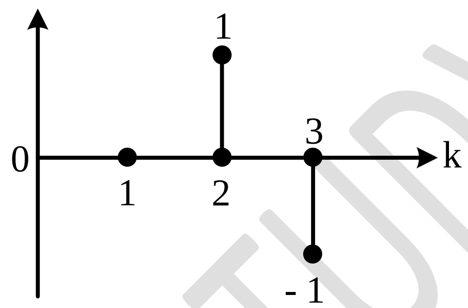
Soln. Given

$h(2) = 1$

$h(3) = -1$

$h(k) = 0$ otherwise

The diagram of the response is as shown



Note, that it has the finite magnitude values. So it is having finite impulse response.

So, FIR

But it is not low pass filter So, S_1 is false

Option (a)

8. The transfer function of a discrete time LTI system is given by

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

Consider the following statements:

S_1 : The system is stable and casual tor

$ROC : \ |z| > \frac{1}{2}$

S_2 : The system is stable but not causal for

$ROC: \ |z| < \frac{1}{4}$

S_3 : The system is neither stable not causal

$For \ \ ROC: \ \frac{1}{4} < |z| < \frac{1}{2}$

Which one of the following statements is valid?

(a) Both S_1 and S_2 are true
(b) Both S_2 and S_3 true
(c) Both S_1 and S_3 are true
(d) S_1, S_2 and S_3 are all true

[GATE 2010: 2 Marks]

Soln. We know:

(i) Causal System : A discrete time LTI system is causal if and only of the ROC of its system function is exterior of a circle, including infinity

(ii) Stable System : A discrete time LTI system is stable if and only if the ROC of its system function includes the unit circle i.e. $|z| = 1$

$$H(z) = \frac{2 - \frac{3}{4} z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}$$

$$= \frac{\left(1 - \frac{1}{4} z^{-1}\right) + \left(1 - \frac{1}{2} z^{-1}\right)}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{1}{2} z^{-1}\right)}$$

$$H(z) = \frac{1}{\left(1 - \frac{1}{4} z^{-1}\right)} + \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)}$$

$$ROC: \ |z| > \frac{1}{2}$$

$$h(n) = \left(\frac{1}{4}\right)^n u(n) + \left(\frac{1}{2}\right)^n u(n)$$

The system is causal, since ROC is exterior of the circle.

ROC of $H(z)$ includes unit circle so it is stable

So, S_1 is true

For ROC : $|z| < \frac{1}{4}$

ROC does not include the unit circle. So system is not stable.

So, S_2 is not true

For ROC : $\frac{1}{4} < |z| < \frac{1}{2}$

ROC does not include unit circle.

So system is not stable. Also ROC is not exterior of $|z| = \frac{1}{2}$. So it not causal

So, S_3 is true

Both S_1 and S_3 are true

9. Two systems $H_1(z)$ *and* $H_2(z)$ are connected in cascade as shown below. The overall output $y(n)$ is the same as the input $x(n)$ with a one unit delay. The transfer function of the second system $H_2(z)$ is

$$x(n) \longrightarrow \boxed{H_1(z) = \frac{(1-0.4z^{1})}{(1-0.6z^{-1})}} \longrightarrow \boxed{H_2(z)} \longrightarrow y(n)$$

(a) $\frac{(1-0.6\,z^{-1})}{z^{-1}(1-0.4\,z^{-1})}$

(b) $\frac{z^{-1}(1-0.6\,z^{-1})}{(1-0.4\,z^{-1})}$

(c) $\frac{z^{-1}(1-0.4\,z^{-1})}{(1-0.6\,z^{-1})}$

(d) $\frac{(1-0.4\,z^{-1})}{z^{-1}(1-0.6\,z^{-1})}$

[GATE 2011: 2 Marks]

Soln. Given

$$y[n] = x[n-1]$$

Taking Z Transfer on both sides

$$Y(z) = z^{-1}\ X(z)$$

$$or\quad \frac{Y(z)}{X(z)} = z^{-1}$$

For cascaded system

$$H(z) = H_1(z).H_2(z)$$

$$z^{-1} = \frac{(1-0.4\,z^{-1})}{(1-0.6\,z^{-1})}\,.H_z(z)$$

Or, $H_2(z) = z^{-1}\frac{(1-0.6\,z^{-1})}{(1-0.4\,z^{-1})}$

Option (b)

10. Let $x[n] = \left(-\frac{1}{9}\right)^n u(n) - \left(-\frac{1}{3}\right)^n u(-n-1)$. The Region of Convergence (ROC) of the Z – Transform of $x[n]$

(a) $is\ |z| > \frac{1}{9}$

(b) $is\ |z| < \frac{1}{3}$

(c) $is\ \frac{1}{3} > |z| > \frac{1}{9}$

(d) $does\ not\ exist$

[GATE 2014: 2 Marks]

Soln. Given

$$x[n] = \left(-\frac{1}{9}\right)^n u(n) - \left(-\frac{1}{3}\right)^n u(-n-1)$$

Let, $x_1(n) = \left(-\frac{1}{9}\right)^n u(n)$ and

$$x_2(n) = \left(-\frac{1}{3}\right)^n u(-n-1)$$

$$x_1(n) = \left(-\frac{1}{9}\right)^n u(n)$$

It is right sided sequence.

$$X_1(z) = \frac{1}{1-\left(-\frac{1}{9}\right)z^{-1}}$$

ROC : $|z| \geq \frac{1}{9}$ or $|z| > \frac{1}{9}$

$$x_2(n) = \left(-\frac{1}{3}\right)^n u(-n-3)$$

It is left sided sequence

$$X_2(z) = \frac{1}{1-\left(-\frac{1}{3}\right)z^{-1}}$$

ROC : $|z| \leq \frac{1}{3}$

When both these transform are added then

ROC must be between

$$\frac{1}{3} > |z| > \frac{1}{9}$$

Option (c)

11. The input-output relationship of a causal stable LTI system is given as

$$y[n] = \alpha\, y[n-1] + \beta\, x[n]$$

If the impulse response $h[n]$ of this system satisfies the condition

$\sum_{n=0}^{\infty} h[n] = 2,$ the relationship between α and β is

(a) $\alpha = 1 - \beta/2$

(b) $\alpha = 1 + \beta/2$

(c) $\alpha = 2\beta$

(d) $\alpha = -2\beta$

[GATE 2014: 2 Marks]

Soln. Given input output relation for causal, stable, LTI system.

$$y[n] = \alpha\, y[n-1] + \beta\, x[n]$$

Taking Z – Transfer

$$Y(z) = \alpha\, z^{-1}\, Y(z) + \beta\, X(z)$$

$$Y(z)[1 - \alpha\, z^{-1}] = \beta\, X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\beta}{1-\alpha\, z^{-1}}$$

$$h(n) = \beta.\alpha^n\, u[n]$$

Also,

$$\sum_{n=0}^{\infty} h(n) = 2 \quad i.e. \quad \sum_{n=0}^{\infty} \beta\, \alpha^n\, u[n] = 2$$

$$\frac{\beta}{1-\alpha} = 2 \quad or \quad \beta = 2 - 2\alpha \quad or \quad \alpha = 1 - {}^{\beta}/_{2}$$

Option (a)

11. The Z – Transform of the sequence $X[n]$ is given by $X(z) = \frac{1}{(1-2z^{-1})^2}$, with the region of convergence $|z| > 2$. Then, $X[2]$ is __________

[GATE 2014: 2 Marks]

Soln. Given

$$X(z) = \frac{1}{(1-2z^{-1})^2} = |z| > 2$$

Or, $X(z) = (1 - 2z^{-1})^{-2}$

$= 1 + 4\,z^{-1} + 12\,z^{-2} + ----$

Expanding by binomial expansion

Taking inverse z – transform

We get, $x[n] = \{1, 4, 12, ---\}$

So, $x[2] = 12$